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A note about late-time wave tails on a dynamical background

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Consider a spherically symmetric spacetime generated by a self-gravitating massless scalar field ϕ and let ψ be a test (nonspherical) massless scalar field propagating on this dynamical background. Gundlach, Price, and Pullin [3] computed numerically the late-time tails for different multipoles of the field ψ and suggested that solutions with compactly supported initial data decay in accord with Price's law as $t^{-(2\ell+3)}$ at timelike infinity. We show that in the case of the time-dependent background Price's law holds only for $\ell = 0$ while for each $\ell \geq 1$ the tail decays as $t^{-(2\ell+2)}$.

The Einstein-massless scalar field system

$$G_{\alpha\beta} = 8\pi \left(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}(\nabla_\mu\phi\nabla^\mu\phi)\right), \qquad g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0\,, \tag{1}$$

restricted to spherical symmetry, has been serving as an important theoretical laboratory for the investigation of nonlinear gravitational phenomena in a rather simple $1+1$ dimensional setting. For this system Christodoulou proved that a generic spherically symmetric solution settles down asymptotically either to Minkowski spacetime (for small data) [1], or to a Schwarzschild black hole (for large data) [2]. The first reliable numerical simulations of the late-time asymptotics of this relaxation process have been done by Gundlach, Price, and Pullin (GPP) [3]. They found that, regardless of the endstate of evolution, the scalar field develops a tail which falls off as t^{-3} near timelike infinity (for compactly supported initial data).

In a recent paper [4] we revisited this problem to emphasize that the asymptotic convergence to a static equilibrium (Minkowski or Schwarzschild) is an essentially nonlinear phenomenon which cannot, despite many assertions to the contrary in the literature, be properly described by the theory of linearized perturbations on a fixed static asymptotically flat background (Price's tails [5, 6]). This is particularly evident for dispersive solutions which asymptote Minkowski spacetime. In that case the quantitative characteristics of the tail (the decay rate and the amplitude) can be obtained using nonlinear perturbation expansion [4]. Since some details of this formal calculation will be needed below, let us now briefly summarize it. In the parametrization

$$ds^2 = \left(1 - \frac{2m(t,r)}{r}\right)^{-1}\left(-e^{2\beta(t,r)}dt^2 + dr^2\right) + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\,, \tag{2}$$

the system (1) takes a particularly convenient form (below an overdot denotes $\partial/\partial t$ and a prime denotes $\partial/\partial r$)

$$m' = 2\pi\, r(r-2m)(\phi'^2 + e^{-2\beta}\dot{\phi}^2)\,, \tag{3}$$

$$\dot{m} = 4\pi\, r(r-2m)\dot{\phi}\,\phi'\,, \tag{4}$$

$$\beta' = \frac{2m}{r(r-2m)}\,, \tag{5}$$

$$\left(e^{-\beta}\dot{\phi}\right)^{\cdot} = \frac{1}{r^2}\left(r^2 e^{\beta}\phi'\right)'\,. \tag{6}$$

Consider small and compactly supported initial data $(\phi, \dot{\phi})_{t=0} = (\varepsilon f(r), \varepsilon g(r))$. Then, up to the order $\mathcal{O}(\varepsilon^3)$, we have

$$\phi = \varepsilon\phi_1 + \varepsilon^3\phi_3, \qquad m = \varepsilon^2 m_2, \qquad \beta = \varepsilon^2\beta_2, \tag{7}$$

where ϕ_1 satisfies the flat-space radial wave equation

$$\Box\phi_1 := \ddot{\phi}_1 - \phi_1'' - \frac{2}{r}\phi_1' = 0, \qquad (\phi_1, \dot{\phi}_1)_{t=0} = (f(r), g(r)), \tag{8}$$

while the second-order perturbations of the metric functions satisfy

$$m_2' = 2\pi r^2(\dot{\phi}_1^2 + \phi_1'^2), \qquad \beta_2' = \frac{2m_2}{r^2}\,. \tag{9}$$

Solving equation (8) and then (9) we get for $t > R$ (where R is the radius of support of initial data)

$$\phi_1(t,r) = \frac{a(t-r)}{r}, \tag{10}$$

$$m_2(t,r) = 2\pi \left(2 \int_{t-r}^{\infty} a'^2(s)\, ds - \frac{a^2(t-r)}{r} \right), \tag{11}$$

$$\beta_2(t,r) = 4\pi \left(-\frac{2}{r} \int_{t-r}^{\infty} a'^2(s)\, ds + 2 \int_{t-r}^{\infty} \frac{a'^2(s)}{t-s}\, ds - \int_{t-r}^{\infty} \frac{a^2(s)}{(t-s)^3}\, ds \right), \tag{12}$$

where the initial-data-generating function $a(u)$ vanishes for $|u| > R$. The third-order perturbation of the scalar field ϕ_3 satisfies the inhomogeneous wave equation (with zero initial data)

$$\Box\phi_3 = 2\beta_2 \ddot{\phi}_1 + \dot{\beta}_2 \dot{\phi}_1 + \beta_2' \phi_1' =: S(t,r)\,. \tag{13}$$

The source $S(t,r)$ is already known from (10-12) so we can use the Duhamel formula

$$\phi_3(t,r) = \frac{1}{2r} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \rho\, S(\tau,\rho)\, d\rho\,, \tag{14}$$

to obtain the asymptotic behavior for large retarded times

$$\phi(t,r) \simeq \varepsilon^3 \phi_3(t,r) \sim \frac{\varepsilon^3 \Gamma_0\, t}{(t^2-r^2)^2}\,, \qquad \Gamma_0 = -2^5\pi \int_{-\infty}^{+\infty} a(u) \int_u^{+\infty} (a'(s))^2 ds\, du\,. \tag{15}$$

We refer the reader to [4] for more details about this calculation and numerical evidence.

After this introduction, we are ready to discuss an interesting model for investigating linear *nonspherical* tails on a fixed dynamical background. This model, proposed by GPP [3], involves a nonspherical test massless scalar field ψ which propagates on the spacetime (2) generated by the self-gravitating field ϕ. Since the dynamics of ψ is linear and the background is spherically symmetric, one may decompose ψ into spherical harmonics

$$\psi(t,r,\vartheta,\varphi) = \sum_{\ell \geq 0, |m| \leq \ell} \psi_{\ell m}(t,r)\, Y_\ell^m(\vartheta,\varphi)\,, \tag{16}$$

and analyze the evolution of each multipole separately

$$\left(e^{-\beta} \dot{\psi}_{\ell m}\right)^{\cdot} - \frac{1}{r^2} \left(r^2 e^{\beta} \psi_{\ell m}'\right)' + e^{\beta} \frac{\ell(\ell+1)}{r(r-2m)} \psi_{\ell m} = 0\,. \tag{17}$$

GPP conjectured[1] that for compactly supported initial data the multipoles have the tail $\psi_{\ell m}(t,r) \sim t^{-(2\ell+3)}$ at timelike infinity ($t \to \infty$ at a fixed r), in accord with Price's law on a fixed Schwarzschild background, even though the actual spherical background is time dependent and its Bondi mass decreases (to a positive value in the collapsing case or to zero in the dispersive case). It seems that GPP's conjecture was based more on belief than numerical evidence, because for the first few multipoles the following power-law exponents of the tail were reported numerically (see Fig.12 in [3]): -2.77 ($\ell = 0$), -3.95 ($\ell = 1$), -5.94 ($\ell = 2$), and -8.34 ($\ell = 3$).

The purpose of this note is to point out that in this model (and for other time-dependent backgrounds) Price's law (*i.e.*, $t^{-(2\ell+3)}$ decay) holds only for $\ell = 0$, while for $\ell \geq 1$ the power-law exponent of the tail is equal to $-(2\ell+2)$ (as was clearly indicated by GPP's own numerics). To show that we shall compute the late-time asymptotic behavior of $\psi_{\ell m}(t,r)$ (for smooth initial data compactly supported in a ball of radius R') along similar lines as described above for the field ϕ. The perturbation expansion has the form $\psi_{\ell m} = \psi_0 + \varepsilon^2 \psi_2 + \ldots$, where for convenience of notation we dropped the multipole indices on iterates. At the zero order we have

$$\Box_{(\ell)} \psi_0 := \ddot{\psi}_0 - \psi_0'' - \frac{2}{r} \psi_0' + \frac{\ell(\ell+1)}{r^2} \psi_0 = 0\,, \qquad (\psi_0, \dot{\psi}_0)_{t=0} = (\psi_{\ell m}, \dot{\psi}_{\ell m})_{t=0}\,, \tag{18}$$

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[1] GPP considered the characteristic initial value problem while we are studying the Cauchy problem, however this difference do not affect the asymptotics of tails.

which for $t > R'$ is solved by

$$\psi_0(t,r) = \frac{1}{r}\sum_{k=0}^{l} \frac{(2\ell-k)!}{k!(\ell-k)!}\frac{b^{(k)}(t-r)}{(2r)^{\ell-k}}\,, \tag{19}$$

where the initial-data-generating function $b(u)$ vanishes for $|u| > R'$ (the superscript in round brackets denotes the k-th derivative). At the second order we get

$$\Box_{(\ell)}\psi_2 = 2\beta_2\ddot{\psi}_0 + \dot{\beta}_2\dot{\psi}_0 + \beta_2'\psi_0' - \frac{2\ell(\ell+1)m_2}{r^3}\psi_0 =: S_\ell(t,r)\,, \qquad (\psi_2,\dot{\psi}_2)_{t=0} = (0,0)\,. \tag{20}$$

Substituting (11), (12), and (19) into the Duhamel formula (where $P_\ell(x)$ is the Legendre polynomial of degree ℓ)

$$\psi_2(t,r) = \frac{1}{2r}\int\limits_0^t d\tau \int\limits_{|t-r-\tau|}^{t+r-\tau} \rho\, P_\ell\left(\frac{r^2+\rho^2-(t-\tau)^2}{2r\rho}\right) S_\ell(\tau,\rho)\, d\rho\,, \tag{21}$$

we get (for large retarded times) for $\ell = 0$:

$$\psi_{00}(t,r) \simeq \varepsilon^2\psi_2(t,r) \sim \frac{\varepsilon^2 B_0\, t}{(t^2-r^2)^2}\,, \qquad B_0 = -2^5\pi \int\limits_{-\infty}^{+\infty} b(u) \int\limits_{u}^{+\infty} (a'(s))^2 ds\, du\,, \tag{22}$$

and for $\ell \geq 1$ (see [7] for technical details of calculation in the $\ell \geq 1$ case):

$$\psi_{\ell m}(t,r) \simeq \varepsilon^2\psi_2(t,r) \sim \frac{\varepsilon^2 B_\ell\, r^\ell}{(t^2-r^2)^{\ell+1}}\,, \qquad B_\ell = (-1)^\ell \frac{2^{\ell+3}\ell!\pi}{2\ell+1}\int\limits_{-\infty}^{+\infty}\left(\frac{\ell^2(\ell-1)}{2\ell-1}a^2(u)b''(u) - 2\ell^2(a'(u))^2 b(u)\right) du\,. \tag{23}$$

The numerical verification of these formulae is summarized in Figure 1 and Table 1.

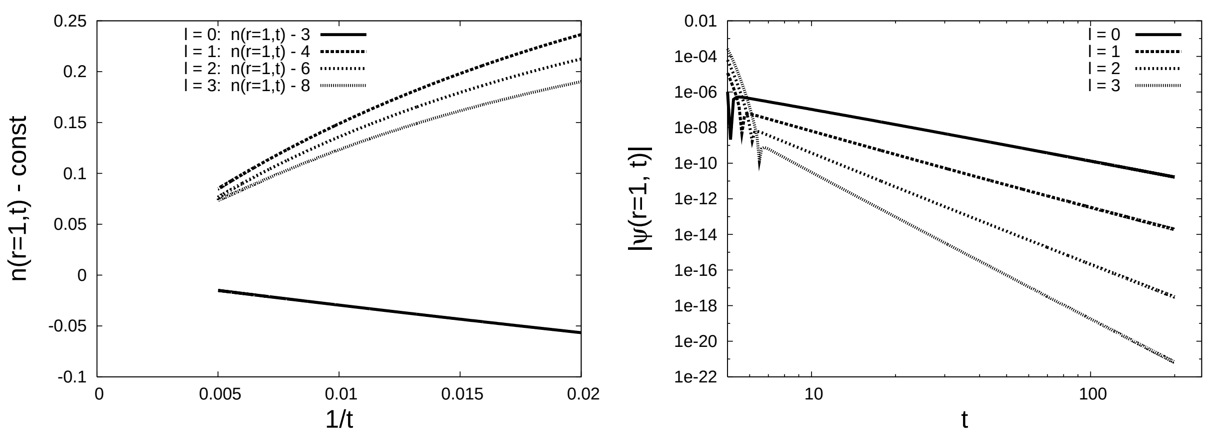

FIG. 1: Left panel: The difference between the local power index and the theoretical prediction: $n(r=1,t) - 3$ for $\ell = 0$ and $n(r=1,t) - 2(\ell+2)$ for $\ell = 1, 2, 3$, as a function of $1/t$. Right panel: The log-log plots of $|\psi(t,\, r=1)|$ for $\ell = 0, 1, 2, 3$. Both panels correspond to initial data generated by $a(x) = \exp(-x^2)/\sqrt{2\pi}$ (for ϕ_1) and $b(x) = x^2\exp(-x^2)$ (for ψ_0), with $\varepsilon = 2^{-8}$.

We fit our numerical data with the formula

$$\psi(t,r) = Bt^{-\gamma}\exp\left(c/t + d/t^2\right)\,, \tag{24}$$

which gives the local power index (LPI) [8]

$$n(t,r) := -t\dot{\psi}(t,r)/\psi(t,r) = \gamma + c/t + 2d/t^2\,. \tag{25}$$

Our fitting procedure proceeds in two steps. First, from the local power index data on the interval $1/t \leq 1/50$ (the left panel of Figure 1) we fit γ, c and d in (25). Next, having determined γ, c and d in this way, we fit B in (24) from ψ data on the interval $50 \leq t$ (the right panel of Figure 1). The results of this procedure are given in Table 1.

ε	Numerics: LPI data			Theory (third order)		Numerics: ψ data
	c	d	γ	γ	B	B
$\ell = 0$						
2^{-12}	-3.068	6.056	3.000	3	-5.296×10^{-4}	-5.293×10^{-7}
2^{-10}	-3.064	5.877	3.000	3	-8.474×10^{-4}	-8.468×10^{-6}
2^{-8}	-3.064	5.867	3.000	3	-1.356×10^{-4}	-1.355×10^{-4}
$\ell = 1$						
2^{-12}	16.73	-134.7	4.008	4	1.084×10^{-7}	1.145×10^{-7}
2^{-10}	16.73	-134.7	4.008	4	1.735×10^{-6}	1.833×10^{-6}
2^{-8}	16.73	-134.7	4.008	4	2.776×10^{-5}	2.932×10^{-5}
$\ell = 2$						
2^{-12}	15.87	-139.4	6.005	6	-6.940×10^{-7}	-7.193×10^{-7}
2^{-10}	15.87	-139.4	6.005	6	-1.110×10^{-5}	-1.151×10^{-5}
2^{-8}	15.87	-139.4	6.005	6	-1.777×10^{-4}	-1.841×10^{-4}
$\ell = 3$						
2^{-12}	13.28	-110.7	8.012	8	6.023×10^{-6}	6.525×10^{-6}
2^{-10}	13.29	-111.0	8.012	8	9.637×10^{-5}	1.044×10^{-4}
2^{-8}	13.31	-111.4	8.012	8	1.542×10^{-3}	1.669×10^{-3}

TABLE I: The comparison of analytic and numerical decay rates and amplitudes of the tails at timelike infinity for $\ell = 0, 1, 2, 3$, for initial data generated by $a(x) = \exp(-x^2)/\sqrt{2\pi}$ (for ϕ_1) and $b(x) = x^2 \exp(-x^2)$ (for ψ_0), at $r = 1$. The theoretical prediction is $B = \varepsilon^2 B_\ell$, with B_0 given in (22) and B_ℓ given in (23) for $\ell \geq 0$. Fits were made on the interval $50 \leq t \leq 200$.

It is instructive to compare the tail (23) with the tail of a massless scalar field propagating on a fixed asymptotically flat *static* background. The latter can be readily obtained from the Duhamel formula (21) applied to the source (20) with $m_2 = M$ and $\beta_2 = -2M/r$, where M is the total mass. The result (valid for all ℓ) reads

$$\psi_{\ell m}(t,r) \sim \frac{C_\ell \, r^\ell t}{(t^2 - r^2)^{\ell+2}}\,, \qquad C_\ell = -M 2^{\ell+3}(\ell+1)! \int_{-\infty}^{+\infty} b(u) du\,. \tag{26}$$

This is the celebrated Price tail [5] (as far as we know, first obtained in the form (26) by Poisson [9]). It is worth stressing that the formula (26) yields a good approximation for the amplitude of the tail provided that both an observer and initial data lie in a weak field region where M/r is small. Note that for $\ell = 0$ the formula (22) takes the form (26) when the total mass M is replaced by the weighted average over the Bondi mass $M(u) = 4\pi \int_u^{+\infty} (a'(s))^2 ds$. For $\ell \geq 1$ Price's tail decays by one power faster than that in (23) which, on a technical level, is due an extra cancelation in the integration by parts of Duhamel's formula.

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